Everything you wanted to know about triangle tiling billiards

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What is a tiling billiard ?

Tiling billiard is a dynamical system that models a movement of light through heterogeneous media, following the Snell's law. Take any tiling of a plane and consider a particle following a straight segment till a moment when it hits a boundary of a tile. Then, it passes to a neighboring tile following the Snell's law with the refraction coefficient $k = -1$! That's the rule !

The materials with $k = -1$ exist in 'nature' (may be constructed as slabs of photonic crystals) and have many rich optical properties, for example to the construction of invisibility cloaks.

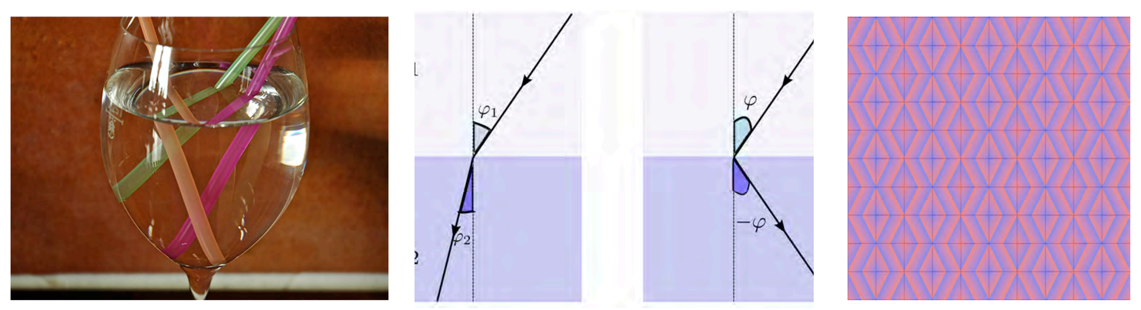


Figure 1:*Snell's law of refraction and how the light ray "breaks".* a. Refraction in water. b. The refraction coefficient $k = -1$, and a foliation of light rays in the square tiling billiard. Here $\frac{\sin\varphi_1}{\sin\varphi_2} = k := \frac{n_1}{n_2}$.

What is this poster about?

Take a periodic triangle tiling obtained by cutting a plane by three families of equidistant parallel lines. The parameters of this tiling are the angles α, β, γ of a tile. A tiling billiard on such a tiling is a **triangle tiling billiard**. Of course, the dynamics depends on the form of a triangle, and on initial conditions. We present here the complete description of the dynamics in triangle tiling billiards, as well as our main tools. This poster is based on our work [3].

Our first tool is geometrical - a triangle tiling billiard has natural *foliations* corresponding to it. The second one is combinatorial - we find a *renormalization process* on such billiards (a way of looking at them under a magnifying glass) which corresponds to *"seeing flowers growing inside the flowers"*. This is just an image... If you want to know what happens more precisely, continue reading.

Some examples of trajectories

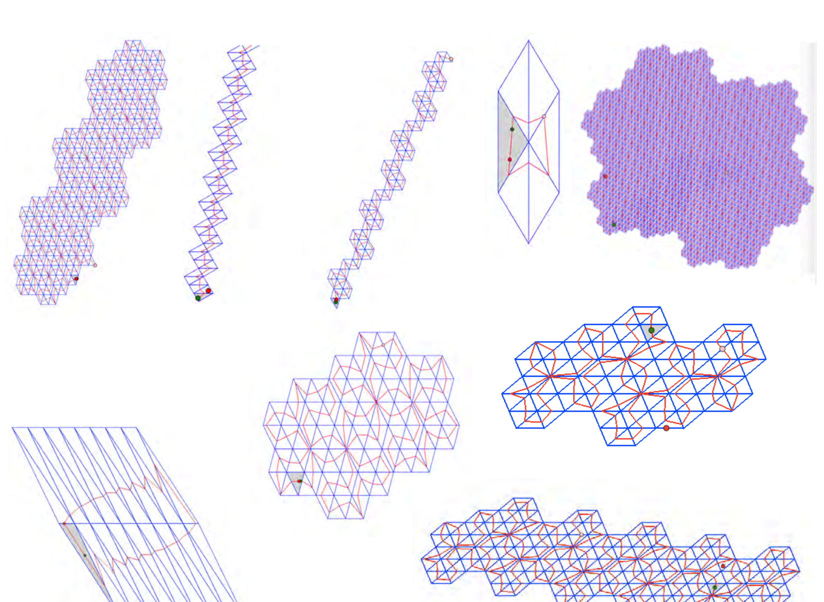


Figure 2:Triangle tiling billiard trajectories: some are periodic, some are linearly escaping. There also exist non-lineraly escaping (**exceptional**) trajectories, passing by all tiles in a tiling. You are maybe surprised that all of the trajectories pass by each tile at most once...

Starting trick: folding

Take two consequent triangles crossed by a trajectory. Then one can fold one of them on another (as a butterfly folds its wings...). A trajectory folds onto itself! This idea from [1] helps to reduce the dynamics of a billiard on a plane to the dynamics of the family CET^3_τ of **interval exchange transformations with flips** on the circle with parameters l_1, l_2, l_3, τ. Here (l_1, l_2, l_3) is a triple of renormalized angles of a tile, and $\tau \in [0, 1]$ corresponds to the position of a trajectory with respect to the *circumcircle of a tile.* Why? Because *all the plane folds into one circle.*

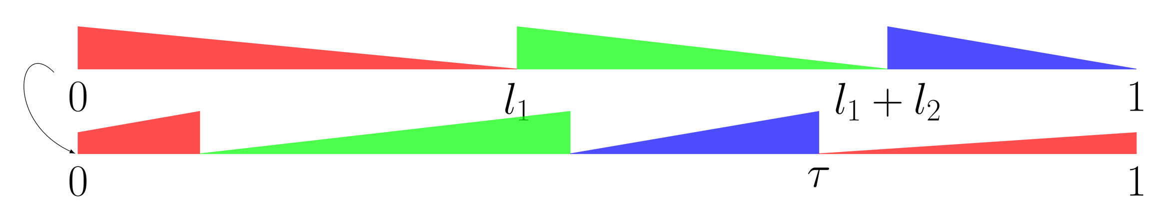


Figure 3:A pictorial representation of a map $F = F^{l_1,l_2,l_3}_\tau \in CET^3_\tau$.

NB: *The dynamics of IETs with flips is different from that of IETs: the last are almost always minimal, and the first are almost never minimal (have periodic intervals).*

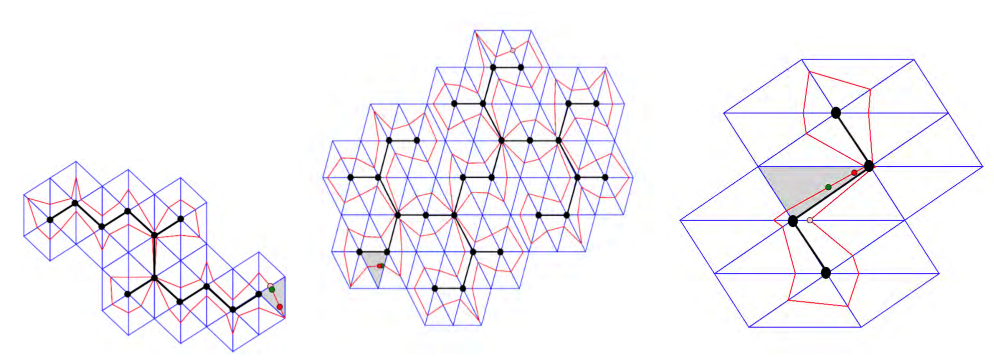
Here is a **corollary** of the folding, proven in [1] for triangle tiling billiards (and generalized in [3] to any *locally foldable tiling*):

1. Any trajectory passes by any tile at most *once*.
2. All bounded trajectories are periodic and stable under perturbation.

First result: the Tree Conjecture holds

Conjecture.[1] Take any periodic trajectory δ of a triangle tiling billiard. Then the vertices and the edges of the tiling that are contoured by δ form a graph which is a *tree*.

This Conjecture has been formulated in [1] and proven for obtuse triangles. We prove it in full generality in [3].



References

The relationship of triangle tiling billiards with IETs with flips was noticed in [1], and some behaviors were conjectured (the Tree and $4n + 2$ Conjectures, and the convergence to the Rauzy fractal in the Tribonacci tiling). In [2], it was proven that the conditions $\tau = \frac{1}{2}$ (passing by circumcenters) and $\rho_\Delta \in \mathcal{R}$ are necessary for the non-linear escape. In [3], we prove also the sufficient direction for the non-linear escape, as well as all the conjectures formulated in [1].

1 P.Baird-Smith, D.Davis, E.Fromm, S.Iyer, *Tiling billiards on triangle tilings, and interval exchange transformations* (2018+)

2 P. Hubert, O. Paris-Romaskevich, *Triangle tiling billiards and the exceptional family of their escaping trajectories* (2019)

3 O. Paris-Romaskevich, *Trees and flowers on a billiard table*, preprint (2019+)

Second result: Classification Theorem

Let $\Delta_2 := \{\boldsymbol{x} = (x_1, x_2, x_3) | x_i \geq 0, \sum_i x_i = 1\} \subset \mathbb{R}^3$. If $x_j > \frac{1}{2}$ for some j, one maps a triple $\boldsymbol{x}$ to a triple $\boldsymbol{x}'$ where $x'_j := 2x_j - 1$ and the other two coordinates $x_i, i \neq j$, stay unchanged. Then we normalize by x_j to get back to Δ_2. This algorithm is called the **Rauzy subtractive algorithm**. The subset $\mathcal{R} \subset \Delta_2$ of triples on which such an algorithm can be applied infinitely, is called the **Rauzy gasket**.It is an *open question* to calculate $\dim_H \mathcal{R}$.

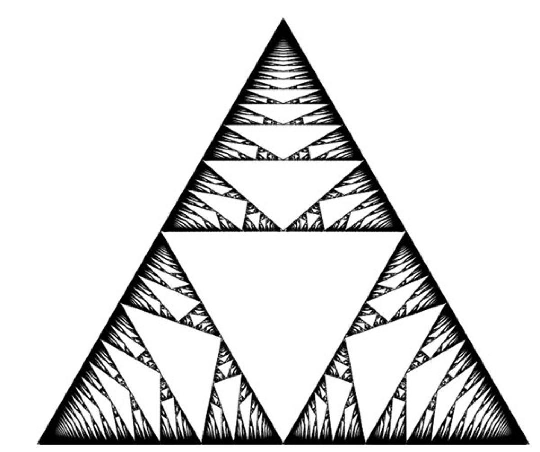


Figure 4:The Rauzy gasket is a fractal, homeomorphic to Sierpinski triangle.

Example. A point $\boldsymbol{a} = (a, a^2, a^3) \in \Delta_2$ with $a + a^2 + a^3 = 1$ belongs to $\mathcal{R}$ since it is a 3-periodic point of the Rauzy subtractive algorithm.

Why we define the set $\mathcal{R}$? It relates to triangle tiling billiards since it characterizes the forms of tilings that *admit non-linearly escaping trajectories*. Let $\rho_\Delta := \left(1 - \frac{2}{\pi}\alpha, 1 - \frac{2}{\pi}\beta, 1 - \frac{2}{\pi}\gamma\right)$.

Classification Theorem for the behavior of triangle tiling billiards. For any (α, β, γ)- triangle tiling billiard, the following holds:

1. if $\rho_\Delta \notin \mathcal{R}$ then any trajectory on a corresponding tiling is either linearly escaping or periodic,
2. if $\rho_\Delta \in \mathcal{R}$ then any trajectory escapes to infinity (is periodic) if and only if it passes (doesn't pass) through a circumcenter of a tile. Moreover, escaping trajectories in this case *pass by all tiles of the tiling*,
3. all of the trajectories on a tiling are periodic if and only if ρ_Δ belongs to the full preimage of the point $(1/3, 1/3, 1/3)$ under the Rauzy subtractive algorithm.

First tool: tiling billiard foliations

Tiling billiard foliations are natural foliations that come from the unfolding a family of parallel chords slicing up the folded plane.

Figure 5:A bird's eye view of a parallel foliation in the Tribonacci tiling, by Ofir David.

Second tool: renormalization

We find a natural renormalization process for the system CET^3_τ. This also defines a renormalization process on triangle tilings.

Theorem. Take a map $F = F^{l_1,l_2,l_3}_\tau \in CET^3_\tau$ with $\tau \in [0, \frac{1}{2}]$. Let $\max\{l_j\}^3_{j=1} \leq \frac{1}{2}$ and $\tau > \max\{l_j\}^3_{j=1}$. Suppose that $l_3 = \min\{l_j\}^3_{j=1}$. Consider the interval $S_3 := (\tau - l_2, l_1 + \tau - l_3) = (s^-_j, s^+_j)$ and reglue its endpoints. Then a first return map on the circle $S_j / s^-_j \sim s^+_j$ is well-defined. Let $R_3 F : \mathbb{S}^1 \to \mathbb{S}^1$ be its rescaling back to the unit circle. Then the parameters of the map $R_3 F = F^{l'_1,l'_2,l'_3}_{\tau'} \in CET^3_\tau$, are given by

$$[l'_1 : l'_2 : l'_3] = [l_1 - l_3 : l_2 - l_3 : l_3],$$
$$\tau' = \frac{1}{2} - r', \quad r' = \frac{r}{|S_3|} \geq r.$$

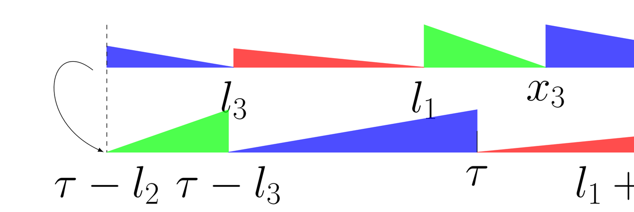


Figure 6:Illustration of the renormalized map $R_3 F$. A miracle here that all of the intervals come back *flipped* and with the same combinatorics.

This renormalization process helps completely describe the symbolic dynamics of triangle tiling billiards (i.e. the sequences of crossed sides). For example, we obtain a following

Corollary. (4n+2 Conjecture from [1])[3] All periodic trajectories have their symbolic codes equal to the squares of some odd-length words in the alphabet of sides $\{a, b, c\}$. Hence, all periods belong to the set $\{4n + 2, n \in \mathbb{N}\}$.

Third result: Tribonacci tiling and the Rauzy fractal

Take such a tiling that $\rho_\Delta = \boldsymbol{a} = (a, a^2, a^3)$. We call this tiling the **Tribonacci tiling.**

Theorem. Any non-singular trajectory δ in the Tribonacci tiling starting in a circumcenter has the following properties:

1. it passes by all tiles;
2. Its parallel foliation contains only periodic trajectories, and these have periods equal to doubled Tribonacci numbers $\{2T_n\}^\infty_{n=4}$, with $T_k = 1, k \leq 3, T_n = T_{n-1} + T_{n-2} + T_{n-3}$,
3. it is approached by growing flowers that *eat each other up* in the same way as is constructed the classical *Rauzy fractal*;
4. it converges to the Rauzy fractal, after reparametrization.

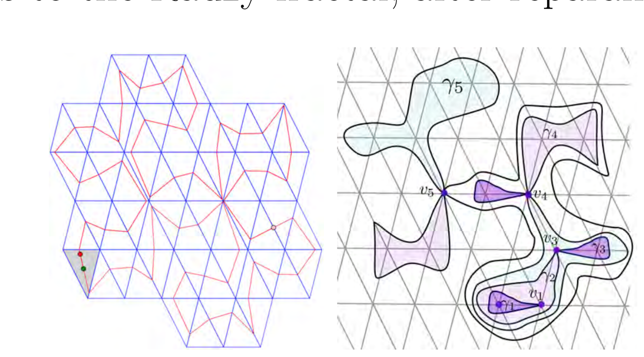


Figure 7:The way the flowers grow in the Tribonacci tiling.

The convergence to the Rauzy fractal was conjectured in [2], as well as (in different terminology) by P. Hooper and B. Weiss while studying real-rel deformations of Arnoux-Yoccoz surface.